Online Supplement to
"On-demand Service Sharing via Collective Dynamic Pricing"

Key words:

1. Proofs with stochastic supply (EC.2.1)

We provide the formulation of the platform's problem as a function of the allocation terms, using a similar expression to Lemma 1. We omit the full formulation and the proof of this result, but these follow from identical arguments as earlier. The first term captures profits when Agent 1 is served before Agent 2 and S2 arrive. The second term captures profits when S2 arrives before Agent 2 ($\omega < \tau$). The third term captures profits instances when Agent 2 arrives before S2 ($\omega > \tau$) and there is no supply shortage. The last term captures profits at time τ when there is a supply shortage.

Lemma 1. *Problem* $(\mathcal{P}_S)$ *is equivalent to:*

$$\max_{\substack{T_1, T_1^\omega, T_1^\tau \\ T_{12}^\tau, T_2^\tau, T_{2S}^\tau}} \quad \widehat{\Pi} = \int_{\underline{\theta}}^{\overline{\theta}} e^{-(r+\lambda+\mu)T_1(\theta_1)} \left(e^{-\delta T_1(\theta_1)}\varphi(\theta_1) - c\right) f(\theta_1) d\theta_1 + \int_{\underline{\theta}}^{\overline{\theta}} \int_0^\infty \mu e^{-(r+\lambda+\mu)\omega} \widehat{\Pi}^\omega(\theta_1) d\omega f(\theta_1) d\theta_1$$
$$+ \int_{\underline{\theta}}^{\overline{\theta}} \int_0^{T_1(\theta_1)} \lambda e^{-(r+\lambda+\mu)\tau} \widehat{\Pi}^\tau(\theta_1) d\tau f(\theta_1) d\theta_1 + \int_{\underline{\theta}}^{\overline{\theta}} \int_{T_1(\theta_1)}^\infty \lambda e^{-(r+\lambda+\mu)\tau} \widehat{\Pi}_{2S}^\tau(\theta_1) d\tau f(\theta_1) d\theta_1$$

subject to monotonicity constraints, where:

$$\widehat{\Pi}^\tau(\theta_1) = \int_{\underline{\theta}}^{\overline{\theta}} e^{-rT_{12}^\tau(\theta_1,\theta_2)} \left(e^{-\delta(\tau + T_{12}^\tau(\theta_1,\theta_2))}\varphi(\theta_1) + e^{-\delta T_{12}^\tau(\theta_1,\theta_2)}\varphi(\theta_2) - c\right) f(\theta_2) d\theta_2$$
$$+ \int_{\underline{\theta}}^{\overline{\theta}} \left[e^{-rT_1^\tau(\theta_1,\theta_2)} \left(e^{-\delta(\tau + T_1^\tau(\theta_1,\theta_2))}\varphi(\theta_1) - c\right) + e^{-rT_2^\tau(\theta_1,\theta_2)} \left(e^{-\delta(T_2^\tau(\theta_1,\theta_2))}\varphi(\theta_2) - c\right)\right] f(\theta_2) d\theta_2.$$

$$\widehat{\Pi}_{2S}^\tau(\theta_1) = \int_\tau^\infty \mu e^{-\mu(\omega-\tau)} \int_{\underline{\theta}}^{\overline{\theta}} e^{-r(\omega + T_2^{\tau,\omega}(\theta_2) - \tau)} \left(e^{-\delta(\omega + T_2^{\tau,\omega}(\theta_2) - \tau)}\varphi(\theta_2) - c\right) f(\theta_2) d\theta_2 d\omega.$$

$$\widehat{\Pi}^\omega(\theta_1) = e^{-(r+\lambda)T_1^\omega(\theta_1)} \left(e^{-\delta(\omega + T_1^\omega(\theta_1))}\varphi(\theta_1) - c\right) + \int_\omega^{\omega + T_1^\omega(\theta_1)} \lambda e^{-(r+\lambda)(\tau-\omega)} \widehat{\Pi}^\tau(\theta_1) d\tau.$$

1.1. Proof of Theorem EC.1

Derivation of $\widehat{\Pi}_{2S}^\tau(\theta_1)$. If there is no supply shortage when Agent 2's arrives, the mechanism unfold as in the baseline setting. Otherwise, the platform serves Agent 2 at time $t = \omega$ if and only if $e^{-\delta(\omega-\tau)}\varphi(\theta_2) \geq c$:

$$T_2^{\tau,\omega}(\theta_2) = \begin{cases} 0, & \text{if } e^{-\delta(\omega-\tau)}\varphi(\theta_2) \geq c, \\ \infty, & \text{if } e^{-\delta(\omega-\tau)}\varphi(\theta_2) < c. \end{cases}$$

This optimal policy will result in the value of $\widehat{\Pi}^{\tau}_{2S}(\theta_1)$:

$$\widehat{\Pi}^{\tau}_{2S}(\theta_1) = \int_{\tau}^{\infty} \mu e^{-(r+\mu)(\omega-\tau)} \int_{\varphi^{-1}(ce^{\delta(\omega-\tau)})}^{\overline{\theta}} \left[e^{-\delta(\omega-\tau)}\varphi(\theta_2) - c\right] f(\theta_2) d\theta_2 d\omega. \tag{1}$$

Derivation of $\widehat{\Pi}^{\tau}(\theta_1)$. At time τ when Agent 2 arrives before S2 and there is no supply shortage, the allocation is identical to the baseline setting. The platform's expected discounted profit is:

$$\widehat{\Pi}^{\tau}(\theta_1) = \begin{cases} \int_{\theta_c}^{\overline{\theta}} (\varphi(\theta_2) - c) f(\theta_2) d\theta_2 & \text{if Agent 1 has been served,} \\ e^{-\delta\tau}\varphi(\theta_1) - c + \int_{\theta_0}^{\overline{\theta}} \varphi(\theta_2) f(\theta_2) d\theta_2 & \text{if Agent 1 is present and } e^{-\delta\tau}\varphi(\theta_1) \geq c, \\ \int_{\varphi^{-1}(c - e^{-\delta\tau}\varphi(\theta_1))}^{\overline{\theta}} (e^{-\delta\tau}\varphi(\theta_1) - c + \varphi(\theta_2)) f(\theta_2) d\theta_2 & \text{if Agent 1 is present and } e^{-\delta\tau}\varphi(\theta_1) < c. \end{cases}$$

For notational convenience, we define:

$$\tilde{\Pi}^{\tau}_{2}(\theta_1) = \int_{\theta_c}^{\overline{\theta}} (\varphi(\theta_2) - c) f(\theta_2) d\theta_2.$$

$$\tilde{\Pi}^{\tau}_{12}(\theta_1) = \begin{cases} e^{-\delta\tau}\varphi(\theta_1) - c + \int_{\theta_0}^{\overline{\theta}} \varphi(\theta_2) f(\theta_2) d\theta_2 & \text{if } e^{-\delta\tau}\varphi(\theta_1) \geq c, \\ \int_{\varphi^{-1}(c - e^{-\delta\tau}\varphi(\theta_1))}^{\overline{\theta}} (e^{-\delta\tau}\varphi(\theta_1) - c + \varphi(\theta_2)) f(\theta_2) d\theta_2 & \text{if } e^{-\delta\tau}\varphi(\theta_1) < c. \end{cases}$$

Derivation of $\widehat{\Pi}^{\omega}(\theta_1)$. At time τ when S2 arrives before Agent 2, the platform chooses $T_1^{\omega}(\theta_1)$ to maximize $\widehat{\Pi}^{\omega}(\theta_1)$. Recall that $T_1^{\omega}(\theta_1) = \infty$ when Agent 1 is not on platform at ω (i.e., if $T_1(\theta_1) < \omega$).

Let us now assume that Agent 1 is on the platform (i.e., if $\omega \leq T_1(\theta_1)$). The problem is equivalent to the platform's problem at time $t = 0$ in the baseline setting (when both suppliers are present on the platform), except that the term $(e^{-\delta T_1(\theta_1)}\varphi(\theta_1) - c)$ is replaced by $(e^{-\delta(\omega + T_1^{\omega}(\theta_1))}\varphi(\theta_1) - c)$ (since Agent 1's virtual value has decayed by a factor $e^{-\delta\omega}$, and time is updated from $t = 0$ to $t = \omega$. Following the same reasoning as in the proof of Theorem 1, we obtain:

$$T_1^{\omega}(\theta_1) = \begin{cases} 0, & \text{if } \omega \leq T_1(\theta_1) \text{ and } e^{-\delta\omega}\varphi(\theta_1) \geq \varphi(\zeta), \\ \infty, & \text{if } \omega \leq T_1(\theta_1) \text{ and } e^{-\delta\omega}\varphi(\theta_1) < \varphi(\zeta), \\ \infty, & \text{if } \omega > T_1(\theta_1). \end{cases}$$

The platform's expected discounted profit is given by:

$$\widehat{\Pi}^{\omega}(\theta_1) = \begin{cases} \int_{\omega}^{\infty} \lambda e^{-(r+\lambda)(\tau-\omega)} \tilde{\Pi}^{\tau}_{2}(\theta_1) d\tau & \text{if Agent 1 has been served,} \\ e^{-\delta\omega}\varphi(\theta_1) - c + \int_{\omega}^{\infty} \lambda e^{-(r+\lambda)(\tau-\omega)} \tilde{\Pi}^{\tau}_{2}(\theta_1) d\tau & \text{if Agent 1 is on the platform and } e^{-\delta\omega}\varphi(\theta_1) \geq \varphi(\zeta), \\ \int_{\omega}^{\infty} \lambda e^{-(r+\lambda)(\tau-\omega)} \tilde{\Pi}^{\tau}_{12}(\theta_1) d\tau. & \text{if Agent 1 is on the platform and } e^{-\delta\omega}\varphi(\theta_1) < \varphi(\zeta). \end{cases}$$

For notational convenience, we define:

$$\tilde{\Pi}^{\omega}_{\emptyset}(\theta_1) = \int_{\omega}^{\infty} \lambda e^{-(r+\lambda)(\tau-\omega)} \tilde{\Pi}^{\tau}_{2}(\theta_1) d\tau$$

$$\tilde{\Pi}^{\omega}_{1}(\theta_1) = \begin{cases} e^{-\delta\omega}\varphi(\theta_1) - c + \int_{\omega}^{\infty} \lambda e^{-(r+\lambda)(\tau-\omega)} \tilde{\Pi}^{\tau}_{2}(\theta_1) d\tau & \text{if } e^{-\delta\omega}\varphi(\theta_1) \geq \varphi(\zeta), \\ \int_{\omega}^{\infty} \lambda e^{-(r+\lambda)(\tau-\omega)} \tilde{\Pi}^{\tau}_{12}(\theta_1) d\tau. & \text{if } e^{-\delta\omega}\varphi(\theta_1) < \varphi(\zeta). \end{cases}$$

Derivation of $T_1(\theta_1)$. We follow the same steps as in the proof of Theorem 1 to show that $T_1(\theta_1)$ is a step function: it is equal to 0 when θ_1 is larger than a threshold $\hat{\zeta}$, and to ∞ otherwise. Let us denote by $\pi(\theta_1)$ the platform's profit contingent on θ_1; from Lemma EC.3, we have

$$\pi(\theta_1) = e^{-(r+\lambda+\mu)T_1(\theta_1)}\left(e^{-\delta T_1(\theta_1)}\varphi(\theta_1) - c\right) + \int_0^\infty \mu e^{-(r+\lambda+\mu)\omega}\widehat{\Pi}^\omega(\theta_1)d\omega$$
$$+ \int_0^{T_1(\theta_1)} \lambda e^{-(r+\lambda+\mu)\tau}\widehat{\Pi}^\tau(\theta_1)d\tau + \int_{T_1(\theta_1)}^\infty \lambda e^{-(r+\lambda+\mu)\tau}\widehat{\Pi}^\tau_{2S}(\theta_1)d\tau.$$

Since any outcome that the platform can achieve with Agent 2 only can also be achieved with both agents, the sum of the last three terms is maximized with $T_1(\theta_1) = \infty$. Moreover, when $\theta_1 < \theta_c$, the first term is also maximized by choosing $T_1(\theta_1) = \infty$ (as in the proof of Theorem 1). Thus, $T_1(\theta_1) = \infty$ for each $\theta_1 \in [\underline{\theta}, \theta_c)$. We henceforth focus on the interval $[\theta_c, \overline{\theta}]$.

CLAIM 1. *For a given $\theta_1 \in [\theta_c, \overline{\theta}]$, let $\overline{T}_{\theta_1} = -\frac{1}{\delta}\log\frac{c}{\varphi(\theta_1)}$. Then the following holds:*

For any $\theta_1 \in [\theta_c, \overline{\theta}]$, if $T_1(\theta_1)$ is finite, then $T_1(\theta_1) < \overline{T}_{\theta_1}$.

The proof is identical to that of Claim 1.

CLAIM 2. *$T_1(\theta_1) = 0$, or $T_1(\theta_1) = \infty$, for each $\theta_1 \in [\underline{\theta}, \overline{\theta}]$.*

Proof of Claim 2. By contradiction, if $T_1(\theta_1) \in (0, \overline{T}_{\theta_1})$, the first-order condition yields:

$$-\left[(r+\lambda+\mu+\delta)e^{-\delta T_1(\theta_1)}\varphi(\theta_1) - (r+\lambda+\mu)c\right]$$
$$+\mu\left(\tilde{\Pi}^\omega_1(\theta_1)\Big|_{\omega=T_1(\theta_1)} - \tilde{\Pi}^\omega_\emptyset(\theta_1)\Big|_{\omega=T_1(\theta_1)}\right) + \lambda\left(\tilde{\Pi}^\tau_{12}(\theta_1)\Big|_{\tau=T_1(\theta_1)} - \widehat{\Pi}^\tau_{2S}(\theta_1)\Big|_{\tau=T_1(\theta_1)}\right) = 0. \quad (2)$$

First, $T_1(\theta_1) < \overline{T}_{\theta_1}$ implies that $e^{-\delta\tau}\varphi(\theta_1) \geq c$. Therefore:

$$\tilde{\Pi}^\tau_{12}(\theta_1)\Big|_{\tau=T_1(\theta_1)} = e^{-\delta T_1(\theta_1)}\varphi(\theta_1) - c + \int_{\theta_0}^{\overline{\theta}} \varphi(\theta_2)f(\theta_2)d\theta_2.$$

Moreover, we know that $T^\omega(\theta_1)\big|_{\omega=T_1(\theta_1)} = 0$. Indeed, since it is optimal to provide an individual service to Agent 1 at time $T_1(\theta_1)$ in the absence of S2, it is also optimal to provide an immediate individual service to them in case S2 arrives at time $\omega = T_1(\theta_1)$. We thus have:

$$\tilde{\Pi}^\omega_1(\theta_1)\Big|_{\omega=T_1(\theta_1)} = e^{-\delta T_1(\theta_1)}\varphi(\theta_1) - c + \int_{T_1(\theta_1)}^\infty \lambda e^{-(r+\lambda)(\tau - T_1(\theta_1))}\tilde{\Pi}^\tau_2(\theta_1)d\tau$$

We obtain from Equation (2):

$$-(r+\delta)e^{-\delta T_1(\theta_1)}\varphi(\theta_1) + rc + \lambda\int_{\theta_0}^{\overline{\theta}} \varphi(\theta_2)f(\theta_2)d\theta_2 - \lambda\widehat{\Pi}^\tau_{2S}(\theta_1)\Big|_{\tau=T_1(\theta_1)} = 0.$$

Recall that $\widehat{\Pi}^{\tau}_{2S}(\theta_1)\Big|_{\tau=T_1(\theta_1)}$ is constant (equal to $\widehat{\Pi}^{\tau}_{2S}$). Therefore, all the terms except the first one are independent from $T_1(\theta_1)$. Since $-e^{-\delta T_1}\varphi(\theta_1)$ is increasing in T_1, we obtain:

$$\frac{\partial \pi(\theta_1)}{\partial T_1}\Big|_{T_1(\theta_1)+\varepsilon} > 0, \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

This contradicts the optimality of $T_1(\theta_1)$: the platform can strictly increase their profit by marginally delaying service to Agent 1. This completes the proof of Claim 2. □

We now know that the optimal value of $T_1(\theta)$ is 0 or ∞. The next step is to determine for which values of θ_1 it is optimal to set $T_1(\theta)=0$ as opposed to $T_1(\theta)=\infty$. We define:

$$\begin{aligned}\Delta(\theta_1) &= \pi(\theta_1)\big|_{T_1(\theta_1)=0} - \pi(\theta_1)\big|_{T_1(\theta_1)=\infty}\\ &= \left(\varphi(\theta_1) - c + \int_0^\infty \mu e^{-(r+\lambda+\mu)\omega}\tilde{\Pi}^{\omega}_{\emptyset}(\theta_1)d\omega + \int_0^\infty \lambda e^{-(r+\lambda+\mu)\tau}\widehat{\Pi}^{\tau}_{2S}(\theta_1)d\tau\right)\\ &\quad - \left(\int_0^\infty \mu e^{-(r+\lambda+\mu)\omega}\tilde{\Pi}^{\omega}_{1}(\theta_1)d\omega + \int_0^\infty \lambda e^{-(r+\lambda+\mu)\tau}\tilde{\Pi}^{\tau}_{12}(\theta_1)d\tau\right),\end{aligned} \tag{3}$$

We show that $\Delta(\theta_1)$ is increasing, which proves that $T_1(\theta_1)$ follows a cutoff rule—there exists $\hat{\zeta}$ such that $T_1(\theta_1)=0$ if $\theta \geq \hat{\zeta}$ and $T_1(\theta_1)=\infty$ otherwise. We also show that $\Delta(\zeta)<0$ so that $\hat{\zeta}>\zeta$.

Claim 3. $\Delta(\theta_1)$ *is strictly increasing with* θ_1.

Proof of Claim 3 Let $\widehat{T}_{\theta_1} \geq 0$ be defined such that $e^{-\delta \widehat{T}_{\theta_1}}\varphi(\theta_1)=\varphi(\zeta)$, for any $\theta_1 \geq \zeta$. For any $\theta_1<\zeta$, we define $\widehat{T}_{\theta_1}=0$. We get:

$$\int_0^\infty \mu e^{-(r+\lambda+\mu)\omega}\tilde{\Pi}^{\omega}_{\emptyset}(\theta_1)d\omega = \int_0^\infty \mu e^{-(r+\lambda+\mu)\omega}\int_\omega^\infty \lambda e^{-(r+\lambda)(\tau-\omega)}\tilde{\Pi}^{\tau}_{2}(\theta_1)d\tau d\omega \tag{4}$$

$$\begin{aligned}\int_0^\infty \mu e^{-(r+\lambda+\mu)\omega}\tilde{\Pi}^{\omega}_{1}(\theta_1)d\omega &= \int_0^{\widehat{T}_{\theta_1}} \mu e^{-(r+\lambda+\mu)\omega}\left(e^{-\delta\omega}\varphi(\theta_1) - c + \int_\omega^\infty \lambda e^{-(r+\lambda)(\tau-\omega)}\tilde{\Pi}^{\tau}_{2}(\theta_1)d\tau\right)d\omega\\ &\quad + \int_{\widehat{T}_{\theta_1}}^\infty \mu e^{-(r+\lambda+\mu)\omega}\int_\omega^\infty \lambda e^{-(r+\lambda)(\tau-\omega)}\tilde{\Pi}^{\tau}_{12}(\theta_1)d\tau d\omega.\end{aligned} \tag{5}$$

Then by plugging Equations (4) and (5) into Equation (3), we get:

$$\begin{aligned}\Delta(\theta_1) &= \varphi(\theta_1) - c + \int_0^\infty \mu e^{-(r+\lambda+\mu)\omega}\int_\omega^\infty \lambda e^{-(r+\lambda)(\tau-\omega)}\tilde{\Pi}^{\tau}_{2}(\theta_1)d\tau d\omega + \int_0^\infty \lambda e^{-(r+\lambda+\mu)\tau}\widehat{\Pi}^{\tau}_{2S}(\theta_1)d\tau\\ &\quad - \int_0^{\widehat{T}_{\theta_1}} \mu e^{-(r+\lambda+\mu)\omega}\left(e^{-\delta\omega}\varphi(\theta_1) - c + \int_\omega^\infty \lambda e^{-(r+\lambda)(\tau-\omega)}\tilde{\Pi}^{\tau}_{2}(\theta_1)d\tau\right)d\omega\\ &\quad - \int_{\widehat{T}_{\theta_1}}^\infty \mu e^{-(r+\lambda+\mu)\omega}\int_\omega^\infty \lambda e^{-(r+\lambda)(\tau-\omega)}\tilde{\Pi}^{\tau}_{12}(\theta_1)d\tau d\omega \quad - \int_0^\infty \lambda e^{-(r+\lambda+\mu)\tau}\tilde{\Pi}^{\tau}_{12}(\theta_1)d\tau.\end{aligned}$$

We write:

$$\varphi(\theta_1) = \int_0^{\widehat{T}_{\theta_1}} \mu e^{-(r+\lambda+\mu)\omega}e^{-\delta\omega}\varphi(\theta_1)d\omega + \int_{\widehat{T}_{\theta_1}}^\infty \mu e^{-(r+\lambda+\mu)\omega}e^{-\delta\omega}\varphi(\theta_1)d\omega + \underbrace{\int_0^\infty (r+\delta+\lambda)e^{-(r+\delta+\lambda+\mu)\omega}\varphi(\theta_1)d\omega}_{=\frac{r+\delta+\lambda}{r+\delta+\lambda+\mu}\varphi(\theta_1)}$$

$$c = \int_0^{\widehat{T}_{\theta_1}} \mu e^{-(r+\lambda+\mu)\omega} c d\omega + \int_{\widehat{T}_{\theta_1}}^{\infty} \mu e^{-(r+\lambda+\mu)\omega} c d\omega + \underbrace{\int_0^{\infty} (r+\lambda) e^{-(r+\lambda+\mu)\omega} c d\omega}_{=\frac{r+\lambda}{r+\lambda+\mu}c}.$$

By re-arranging the terms, we obtain:

$$\Delta(\theta_1) = \frac{r+\delta+\lambda}{r+\delta+\lambda+\mu}\varphi(\theta_1) - \underbrace{\frac{r+\lambda}{r+\lambda+\mu}c}_{\text{Constant}} + \underbrace{\int_0^{\infty} \lambda e^{-(r+\lambda+\mu)\tau} \widehat{\Pi}_{2S}^{\tau}(\theta_1) d\tau}_{\text{Constant}} - \int_0^{\infty} \lambda e^{-(r+\lambda+\mu)\tau} \tilde{\Pi}_{12}^{\tau}(\theta_1) d\tau$$

$$+ \int_{\widehat{T}_{\theta_1}}^{\infty} \mu e^{-(r+\lambda+\mu)\omega} \underbrace{\left[e^{\delta\omega}\varphi(\theta_1) - c + \int_{\omega}^{\infty} \lambda e^{-(r+\lambda)(\tau-\omega)} \tilde{\Pi}_2^{\tau}(\theta_1) d\tau - \int_{\omega}^{\infty} \lambda e^{-(r+\lambda)(\tau-\omega)} \tilde{\Pi}_{12}^{\tau}(\theta_1) d\tau \right]}_{=\Delta^{\omega}(\theta_1)} d\omega$$

We first show that $\frac{r+\delta+\lambda}{r+\delta+\lambda+\mu}\varphi(\theta_1) - \int_0^{\infty} \lambda e^{-(r+\lambda+\mu)\tau} \tilde{\Pi}_{12}^{\tau}(\theta_1) d\tau$ is non-decreasing in θ_1. By following the exact same procedure as earlier (by replacing the term λ by the term $\lambda+\mu$ in the exponential functions and in the denominators), we have:

$$\frac{r+\delta+\lambda}{r+\delta+\lambda+\mu}\varphi(\theta_1) - \int_0^{\infty} \lambda e^{-(r+\lambda+\mu)\tau} \tilde{\Pi}_{12}^{\tau}(\theta_1) d\tau = \text{Constant} + \frac{r+\delta}{r+\delta+\lambda+\mu}\varphi(\theta_1)$$

$$+ \int_{\overline{T}_{\theta_1}}^{\infty} \lambda e^{-(r+\lambda+\mu)\tau} Y^{\tau}(\theta_1) d\tau - \frac{\delta\lambda e^{-(r+\lambda+\mu)\overline{T}_{\theta_1}} F(\theta_0)}{(r+\lambda+\mu)(r+\delta+\lambda+\mu)} c,$$

where $Y^{\tau}(\theta_1)$ is defined as in the proof of Theorem 1:

$$Y^{\tau}(\theta_1) = \int_{\theta_0}^{\varphi^{-1}(c - e^{-\delta\tau}\varphi(\theta_1))} \left(e^{-\delta\tau}\varphi(\theta_1) - c + \varphi(\theta_2) \right) f(\theta_2) d\theta_2.$$

We know that $\varphi(\theta_1)$ and $\overline{T}_{\theta_1}$ are non-decreasing in θ_1. Moreover, using the exact same arguments as in the proof of Theorem 1, we also know that $\int_{\overline{T}_{\theta_1}}^{\infty} \lambda e^{-(r+\lambda+\mu)\tau} Y^{\tau}(\theta_1) d\tau$ is non-decreasing in θ_1. This proves that $\frac{r+\delta+\lambda}{r+\delta+\lambda+\mu}\varphi(\theta_1) - \int_0^{\infty} \lambda e^{-(r+\lambda+\mu)\tau} \tilde{\Pi}_{12}^{\tau}(\theta_1) d\tau$ is non-decreasing in θ_1.

Let us now prove that $\int_{\widehat{T}_{\theta_1}}^{\infty} \mu e^{-(r+\lambda+\mu)\omega} \Delta^{\omega}(\theta_1) d\omega$ is non-decreasing in θ_1. This is clearly satisfied when $\theta_1 < \zeta$ since for such values of θ_1 we have $\widehat{T}(\theta_1) = 0$, and we know that $\Delta^{\omega}(\theta_1)$ is increasing in θ_1. We now prove it for $\theta_1 \geq \zeta$.

1. For any $\theta_1, \theta_1' \in [\zeta, \overline{\theta}]$, we have $\overline{T}_{\theta_1} - \widehat{T}_{\theta_1} = \overline{T}_{\theta_1'} - \widehat{T}_{\theta_1'} \geq 0$. This directly comes from the following definitions $e^{-\delta\overline{T}_{\theta_1}}\varphi(\theta_1) = e^{-\delta\overline{T}_{\theta_1'}}\varphi(\theta_1') = c$ and $e^{-\delta\widehat{T}_{\theta_1}}\varphi(\theta_1) = e^{-\delta\widehat{T}_{\theta_1'}}\varphi(\theta_1') = \varphi(\zeta)$.
2. For any $\theta_1, \theta_1' \in [\zeta, \overline{\theta}]$, we prove that $\Delta^{\omega}(\theta_1) = \Delta^{\omega - (\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'})}(\theta_1')$. Recall that:

$$\Delta^{\omega}(\theta_1) = e^{-\delta\omega}\varphi(\theta_1) \underbrace{- c + \int_0^{\infty} \lambda e^{-(r+\lambda)\tau} \tilde{\Pi}_2^{\tau}(\theta_1) d\tau}_{\text{Constant}} - \int_{\omega}^{\infty} \lambda e^{-(r+\lambda)(\tau-\omega)} \tilde{\Pi}_{12}^{\tau}(\theta_1) d\tau.$$

By definition of $\widehat{T}_{\theta_1}$, $e^{-\delta(\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'})}\varphi(\theta_1) = \varphi(\theta_1')$. Moreover, we prove that, for every $\tau > 0$:

$$\tilde{\Pi}_{12}^{\tau}(\theta_1) = \tilde{\Pi}_{12}^{\tau - (\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'})}(\theta_1').$$

To see this, recall that, for every $\tau \geq 0$ and every $\theta_1 \in [\underline{\theta}, \overline{\theta_1}]$:

$$\tilde{\Pi}_{12}^{\tau}(\theta_1) = \begin{cases} e^{-\delta\tau}\varphi(\theta_1) - c + \int_{\theta_0}^{\overline{\theta}} \varphi(\theta_2) f(\theta_2) d\theta_2 & \text{if } \tau < \overline{T}_{\theta_1} \\ \int_{\varphi^{-1}(c - e^{-\delta\tau}\varphi(\theta_1))}^{\overline{\theta}} \left(e^{-\delta\tau}\varphi(\theta_1) - c + \varphi(\theta_2)\right) f(\theta_2) d\theta_2 & \text{if } \tau \geq \overline{T}_{\theta_1} \end{cases}$$

If $\tau < \overline{T}_{\theta_1}$, which implies that $\tau - (\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'}) < \overline{T}_{\theta_1'}$, we have:

$$\tilde{\Pi}_{12}^{\tau}(\theta_1) = e^{-\delta\tau}\varphi(\theta_1) - c + \int_{\theta_0}^{\overline{\theta}} \varphi(\theta_2) f(\theta_2) d\theta_2 = e^{-\delta\tau} e^{\delta(\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'})} \varphi(\theta_1') - c + \int_{\theta_0}^{\overline{\theta}} \varphi(\theta_2) f(\theta_2) d\theta_2 = \tilde{\Pi}_{12}^{\tau - (\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'})}(\theta_1)$$

If $\tau \geq \overline{T}_{\theta_1}$, which implies that $\tau - (\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'}) \geq \overline{T}_{\theta_1'}$, we have:

$$\tilde{\Pi}_{12}^{\tau}(\theta_1) = \int_{\varphi^{-1}(c - e^{-\delta\tau}\varphi(\theta_1))}^{\overline{\theta}} \left(e^{-\delta\tau}\varphi(\theta_1) - c + \varphi(\theta_2)\right) f(\theta_2) d\theta_2 = \int_{\theta_0}^{\overline{\theta}} \left(e^{-\delta\tau}\varphi(\theta_1) - c + \varphi(\theta_2)\right) f(\theta_2) d\theta_2 - Y^{\tau}(\theta_1)$$

We know from the proof of Theorem 1 that $Y^{\tau}(\theta_1') = Y^{\tau + \left(\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'}\right)}(\theta_1)$. Therefore:

$$\tilde{\Pi}_{12}^{\tau}(\theta_1) = \int_{\theta_0}^{\overline{\theta}} \left(e^{-\delta\tau} e^{\delta(\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'})} \varphi(\theta_1') - c + \varphi(\theta_2)\right) f(\theta_2) d\theta_2 - Y^{\tau - (\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'})}(\theta_1) = \tilde{\Pi}_{12}^{\tau - (\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'})}(\theta_1')$$

As a result, we have:

$$\begin{aligned} \Delta^{\omega}(\theta_1) &= \text{Constant} + e^{-\delta\omega} e^{\delta(\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'})} \varphi(\theta_1') - \int_{\omega}^{\infty} \lambda e^{-(r+\lambda)(\tau - \omega)} \tilde{\Pi}_{12}^{\tau - (\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'})}(\theta_1') d\tau \\ &= \text{Constant} + e^{-\delta\omega} e^{\delta(\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'})} \varphi(\theta_1') - \int_{\omega - (\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'})}^{\infty} \lambda e^{-(r+\lambda)(\tau + (\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'}) - \omega)} \tilde{\Pi}_{12}^{\tau}(\theta_1') d\tau \\ &= \Delta^{\omega - (\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'})}(\theta_1') \end{aligned}$$

3. We conclude that $\int_{\widehat{T}_{\theta_1}}^{\infty} \mu e^{-(r+\lambda+\mu)\omega} \Delta^{\omega}(\theta_1) d\omega$ is increasing in θ_1. Let us consider $\theta_1, \theta_1' \in [\zeta, \overline{\theta}]$ such that $\theta_1 > \theta_1'$. We have:

$$\begin{aligned} \int_{\widehat{T}_{\theta_1}}^{\infty} \mu e^{-(r+\lambda+\mu)\omega} \Delta^{\omega}(\theta_1) d\omega &= \int_{\widehat{T}_{\theta_1}}^{\infty} \mu e^{-(r+\lambda+\mu)\omega} \Delta^{\omega - (\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'})}(\theta_1') d\omega \\ &= \underbrace{e^{-(r+\lambda+\mu)(\widehat{T}_{\theta_1} - \widehat{T}_{\theta_1'})}}_{<1} \underbrace{\int_{\widehat{T}_{\theta_1'}}^{\infty} \mu e^{-(r+\lambda+\mu)\omega} \Delta^{\omega}(\theta_1') d\omega}_{\leq 0 \text{ because } \Delta^{\omega}(\theta_1') \leq 0 \text{ for all } \omega \geq \widehat{T}_{\theta_1'}} \\ &> \int_{\widehat{T}_{\theta_1'}}^{\infty} \mu e^{-(r+\lambda+\mu)\omega} \Delta^{\omega}(\theta_1') d\omega. \end{aligned}$$

This completes the proof of Claim 3. □

CLAIM 4. *It holds that $\hat{\zeta} \geq \zeta$.*

Proof of Claim 4 It is sufficient to show that $\Delta(\zeta) <= 0$, that is, all else equal, the benefit of serving Agent 1 immediately is lower when the second supplier has not arrived yet than when there are two suppliers on the market. By definition of ζ, it is optimal to set $T_1^\omega(\theta_1)\big|_{\omega=0} = \infty$ for all $\theta_1 < \zeta$. As a result, for all $\theta_1 < \zeta$, it is optimal to set $T_1^\omega(\theta_1) = \infty$ for all $\omega > 0$. Hence:

$$\tilde{\Pi}_\emptyset^\omega(\theta_1) = \int_\omega^\infty \lambda e^{-(r+\lambda)(\tau-\omega)} \tilde{\Pi}_2^\tau(\theta_1) d\tau$$
$$\tilde{\Pi}_1^\omega(\zeta) = \int_\omega^\infty \lambda e^{-(r+\lambda)(\tau-\omega)} \tilde{\Pi}_{12}^\tau(\zeta) d\tau$$

From Equation (3), it comes:

$$\Delta(\zeta) = \varphi(\zeta) - c + \int_0^\infty \mu e^{-(r+\lambda+\mu)\omega} \int_\omega^\infty \lambda e^{-(r+\lambda)(\tau-\omega)} \tilde{\Pi}_2^\tau(\zeta) d\tau d\omega + \int_0^\infty \lambda e^{-(r+\lambda+\mu)\tau} \widehat{\Pi}_{2S}^\tau(\zeta) d\tau$$
$$- \int_0^\infty \mu e^{-(r+\lambda+\mu)\omega} \int_\omega^\infty \lambda e^{-(r+\lambda)(\tau-\omega)} \tilde{\Pi}_{12}^\tau(\zeta) d\tau d\omega - \int_0^\infty \lambda e^{-(r+\lambda+\mu)\tau} \tilde{\Pi}_{12}^\tau(\zeta) d\tau.$$

By developing the double integrals, we obtain:

$$\int_{\omega=0}^\infty \mu e^{-(r+\lambda+\mu)\omega} \int_{\tau=\omega}^\infty \lambda e^{-(r+\lambda)(\tau-\omega)} \tilde{\Pi}_{12}^\tau(\zeta) d\tau d\omega = \int_{\tau=0}^\infty \lambda e^{-(r+\lambda)\tau} \int_{\omega=0}^\tau \mu e^{-\mu\omega} \tilde{\Pi}_{12}^\tau(\zeta) d\omega d\tau$$
$$= \int_0^\infty \lambda e^{-(r+\lambda)\tau} \tilde{\Pi}_{12}^\tau(\zeta) d\tau - \int_0^\infty \lambda e^{-(r+\lambda+\mu)\tau} \tilde{\Pi}_{12}^\tau(\zeta) d\tau$$
$$\int_{\omega=0}^\infty \mu e^{-(r+\lambda+\mu)\omega} \int_{\tau=\omega}^\infty \lambda e^{-(r+\lambda)(\tau-\omega)} \tilde{\Pi}_2^\tau(\zeta) d\tau d\omega = \int_0^\infty \lambda e^{-(r+\lambda)\tau} \tilde{\Pi}_2^\tau(\zeta) d\tau - \int_0^\infty \lambda e^{-(r+\lambda+\mu)\tau} \tilde{\Pi}_2^\tau(\zeta) d\tau.$$

It yields:

$$\Delta(\zeta) = \varphi(\zeta) - c + \int_0^\infty \lambda e^{-(r+\lambda)\tau} \left(\tilde{\Pi}_2^\tau(\zeta) - \tilde{\Pi}_{12}^\tau(\zeta) \right) d\tau + \int_0^\infty \lambda e^{-(r+\lambda+\mu)\tau} \left(\widehat{\Pi}_{2S}^\tau(\zeta) d\tau - \tilde{\Pi}_2^\tau(\zeta) \right) d\tau$$
$$= + \int_0^\infty \lambda e^{-(r+\lambda+\mu)\tau} \left(\widehat{\Pi}_{2S}^\tau(\zeta) d\tau - \tilde{\Pi}_2^\tau(\zeta) \right) d\tau$$
$$< \Delta^0(\zeta).$$

The second equality stems from the fact that the platform is indifferent between serving or delaying Agent 1 when both suppliers are present. The last inequality stems from the fact that the platform's future expected profit at time τ is larger when the second supplier has already arrived on the platform. Formally:

$$\widehat{\Pi}_{2S}^\tau(\zeta) = \int_\tau^\infty \mu e^{-(r+\mu)(\omega-\tau)} \int_{\varphi^{-1}(ce^{\delta(\omega-\tau)})}^{\overline{\theta}} \left[e^{-\delta(\omega-\tau)} \varphi(\theta_2) - c \right] f(\theta_2) d\theta_2 d\omega$$
$$\leq \int_\tau^\infty \mu e^{-(r+\mu)(\omega-\tau)} \int_{\theta_c}^{\overline{\theta}} \left[e^{-\delta(\omega-\tau)} \varphi(\theta_2) - c \right] f(\theta_2) d\theta_2 d\omega \quad \text{(because } ce^{\delta(\omega-\tau)} \geq c\text{)}$$
$$< \int_\tau^\infty \mu e^{-(r+\mu)(\omega-\tau)} \int_{\theta_c}^{\overline{\theta}} \left[\varphi(\theta_2) - c \right] f(\theta_2) d\theta_2 d\omega$$
$$= \frac{\mu}{r+\mu} \int_{\theta_c}^{\overline{\theta}} \left[\varphi(\theta_2) - c \right] f(\theta_2) d\theta_2$$

$$< \int_{\theta_c}^{\overline{\theta}} [\varphi(\theta_2) - c]\, f(\theta_2) d\theta_2$$
$$= \tilde{\Pi}_2^{\tau}(\zeta).$$

This completes the proof of Claim 4. □

Finally, this allocation rule satisfies the monotonicity constraints, and is therefore optimal. This completes the proof of Theorem EC.1. □

2. Proofs with disutility and higher cost for sharing (EC.2.2)

Note, first, that Assumption 2 provides a necessary condition to provide shared services; otherwise, each agent receives an individual service with no delay if their type exceeds θ_c, or leaves the platform unserved otherwise. To see this, let A_1 and A_2 be the discounted virtual values of Agent 1 and Agent 2, respectively, at any point in time. A shared service contributes $\gamma(A_1 + A_2) - (1+\alpha)c$ to the platform's objective, whereas two separate services contribute $A_1 + A_2 - 2c$; if $\gamma \leq \frac{1+\alpha}{2}$, we always have $A_1 + A_2 - 2c > \gamma(A_1 + A_2) - (1+\alpha)c$. Therefore, if Assumption 2 is violated, no shared services are provided and each agent receives an immediate individual service if and only if their type exceeds θ_c.

2.1. Proof of Theorem EC.2

Let us assume that $\theta_1 \geq \theta_c$ (we treat the other case similarly at the end of the proof). In that case, the virtual type of Agent 1 exceeds the cost of an individual service.

Let us first consider the service at time τ when both agents are present on the platform (i.e., $\tau \leq T_1(\theta_1)$). If $e^{-\delta\tau}\varphi(\theta_1) \geq c$, Agent 1 will receive a service at time τ, leading to three possibilities: (i) an individual service to Agent 1, with profit contribution $e^{-\delta\tau}\varphi(\theta_1) - c$; (ii) a shared service, with profit contribution $\gamma\left(e^{-\delta\tau}\varphi(\theta_1) + \varphi(\theta_2)\right) - (1+\alpha)c$; and (iii) separate individual services to both agents, with profit contribution $e^{-\delta\tau}\varphi(\theta_1) + \varphi(\theta_2) - 2c$. Note that (i) does not depend on θ_2, (ii) increases with θ_2 at rate $\gamma < 1$, and (iii) increases with θ_2 at rate 1. Expressions (i) and (ii) coincide when $\varphi(\theta_2) = \frac{(1-\gamma)e^{-\delta\tau}\varphi(\theta_1)}{\gamma} + \frac{\alpha c}{\gamma}$. Expressions (ii) and (iii) coincide when $\varphi(\theta_2) = \frac{(1-\alpha)c}{1-\gamma} - e^{-\delta\tau}\varphi(\theta_1)$. Therefore, service provision is given as follows as long as $\frac{(1-\gamma)e^{-\delta\tau}\varphi(\theta_1)}{\gamma} + \frac{\alpha c}{\gamma} < \frac{(1-\alpha)c}{1-\gamma} - e^{-\delta\tau}\varphi(\theta_1)$:

$$\begin{cases} \text{(i) individual service to Agent 1 if } \varphi(\theta_2) < \frac{(1-\gamma)e^{-\delta\tau}\varphi(\theta_1)}{\gamma} + \frac{\alpha c}{\gamma} \\ \text{(ii) shared service if } \frac{(1-\gamma)e^{-\delta\tau}\varphi(\theta_1)}{\gamma} + \frac{\alpha c}{\gamma} \leq \varphi(\theta_2) < \frac{(1-\alpha)c}{1-\gamma} - e^{-\delta\tau}\varphi(\theta_1) \\ \text{(iii) separate individual services otherwise.} \end{cases}$$

To complete this proof, we need to show that $\frac{(1-\gamma)e^{-\delta\tau}\varphi(\theta_1)}{\gamma} + \frac{\alpha c}{\gamma} < \frac{(1-\alpha)c}{1-\gamma} - e^{-\delta\tau}\varphi(\theta_1)$ when Agent 1 remains on the platform at time τ. Assuming otherwise, we would have $e^{-\delta\tau}\varphi(\theta_1) \geq \frac{(\gamma-\alpha)c}{1-\gamma}$, which implies that $\varphi(\theta_1) > \frac{(\gamma-\alpha)c}{1-\gamma} > \frac{(1-\alpha)c}{2(1-\gamma)} > 0$ (Assumption 2). Then, providing an individual service to Agent 1 at time 0 dominates leaving them unserved. Next, we show that a shared service is dominated even if Agent 2 arrives instantaneously, at time 0. Indeed, there are three possibilities: (i) an individual service to Agent 1, with profit contribution $\varphi(\theta_1)-c$; (ii) a shared service, with profit contribution $\gamma\left(\varphi(\theta_1)+\varphi(\theta_2)\right)-(1+\alpha)c$; and (iii) separate individual services to both agents, with profit contribution $\varphi(\theta_1)+\varphi(\theta_2)-2c$. Expression (i) dominates Expression (ii) when $\varphi(\theta_2)<\frac{1-\gamma}{\gamma}\varphi(\theta_1)+\frac{\alpha c}{\gamma}$; and Expression (iii) dominates Expression (ii) when $\varphi(\theta_2)>\frac{(1-\alpha)c}{1-\gamma}-\varphi(\theta_1)$. But since $\varphi(\theta_1)>\frac{(\gamma-\alpha)c}{1-\gamma}$, we have $\frac{1-\gamma}{\gamma}\varphi(\theta_1)+\frac{\alpha c}{\gamma}>\frac{(1-\alpha)c}{1-\gamma}-\varphi(\theta_1)$. Therefore, a shared service at $\tau=0$ is dominated by an allocation policy in which agent 1 is served individually, and so is a shared service at $\tau>0$ as a result. This proves that $\frac{(1-\gamma)e^{-\delta\tau}\varphi(\theta_1)}{\gamma}+\frac{\alpha c}{\gamma}<\frac{(1-\alpha)c}{1-\gamma}-e^{-\delta\tau}\varphi(\theta_1)$ whenever Agent 1 remains on the platform at time $\tau>0$, and completes the proof of the above allocation rule.

We proceed similarly when $e^{-\delta\tau}\varphi(\theta_1)<c$. There are three possibilities: (i) no service, with profit contribution 0; (ii) a shared service, with profit contribution $\gamma\left(e^{-\delta\tau}\varphi(\theta_1)+\varphi(\theta_2)\right)-(1+\alpha)c$; (iii) an individual service to only agent 2, with profit contribution $\varphi(\theta_2)-c$. Expressions (i) and (ii) coincide when $\varphi(\theta_2)=\frac{1+\alpha}{\gamma}c-e^{-\delta\tau}\varphi(\theta_1)$; and Expressions (ii) and (iii) coincide when $\varphi(\theta_2)=\frac{\gamma}{1-\gamma}e^{-\delta\tau}\varphi(\theta_1)-\frac{\alpha}{1-\gamma}c$. When $e^{-\delta\tau}\varphi(\theta_1)\geq\frac{1+\alpha-\gamma}{\gamma}c$, we have $\frac{1+\alpha}{\gamma}c-e^{-\delta\tau}\varphi(\theta_1)\leq\frac{\gamma}{1-\gamma}e^{-\delta\tau}\varphi(\theta_1)-\frac{\alpha}{1-\gamma}c$ holds; otherwise, Expression (ii) is never optimal so Agents 1 and 2 are treated independently. Therefore, service provision is given as follows

$$\text{If } e^{-\delta\tau}\varphi(\theta_1)\geq\frac{1+\alpha-\gamma}{\gamma}c\text{:}\begin{cases}\text{(i) no service if } \varphi(\theta_2)<\frac{1+\alpha}{\gamma}c-e^{-\delta\tau}\varphi(\theta_1)\\ \text{(ii) shared service if } \frac{1+\alpha}{\gamma}c-e^{-\delta\tau}\varphi(\theta_1)\leq\varphi(\theta_2)<\frac{\gamma}{1-\gamma}e^{-\delta\tau}\varphi(\theta_1)-\frac{\alpha}{1-\gamma}c\\ \text{(iii) individual service to Agent 2 otherwise.}\end{cases}$$

$$\text{If } e^{-\delta\tau}\varphi(\theta_1)<\frac{1+\alpha-\gamma}{\gamma}c\text{:}\begin{cases}\text{(i) no service if } \varphi(\theta_2)<c\\ \text{(iii) individual service to Agent 2 otherwise.}\end{cases}$$

We now characterize service provision to Agent 1 at time 0, by deriving $T_1(\theta_1)$. We already know that $T_1(\theta_1)=0$ when $\varphi(\theta_1)\geq\frac{\gamma-\alpha}{1-\gamma}c$. Let us now assume that $\varphi(\theta_1)<\frac{\gamma-\alpha}{1-\gamma}c$. We denote by $\tilde{\Pi}^\tau(\theta_1|\,T_1(\theta_1))$ the platform's expected discounted profit from time $t=\tau$ onward. When $\tau\leq T_1(\theta_1)$, both agents are present on platform at τ, and we denote the corresponding value of $\tilde{\Pi}^\tau(\theta_1|\,T_1(\theta_1))$ by $\tilde{\Pi}^\tau_{12}(\theta_1)$. When $\tau>T_1(\theta_1)$, only Agent 2 is present on platform at τ, and we denote the corresponding value of $\tilde{\Pi}^\tau(\theta_1|\,T_1(\theta_1))$ by $\tilde{\Pi}^\tau_2(\theta_1)$. For a given report θ_1, let $\pi(\theta_1)$ denote the platform's profit:

$$\pi(\theta_1)=e^{-(r+\lambda)T_1(\theta_1)}\left(e^{-\delta T_1(\theta_1)}\varphi(\theta_1)-c\right)+\int_0^{T_1(\theta_1)}\lambda e^{-(r+\lambda)\tau}\tilde{\Pi}^\tau_{12}(\theta_1)+\int_{T_1(\theta_1)}^{\infty}\lambda e^{-(r+\lambda)\tau}\tilde{\Pi}^\tau_2(\theta_1).$$

Claim 5. *For a given* $\theta_1\in[\theta_c,\overline{\theta}]$, *let* $\overline{T}_{\theta_1}=-\frac{1}{\delta}\log\frac{c}{\varphi(\theta_1)}$. *If* $T_1(\theta_1)$ *is finite, then* $T_1(\theta_1)<\overline{T}_{\theta_1}$.

Claim 6. $T_1(\theta_1)=0$, *or* $T_1(\theta_1)=\infty$, *for each* $\theta_1\in[\underline{\theta},\overline{\theta}]$.

These claims are proved as Claims 1 and 2, except that the profit equation associated with shared services at time τ needs to be modified to account for the added costs and disutility. Specifically, we replace Equation EC.3 by

$$\tilde{\Pi}^\tau_{12}(\theta_1)\Big|_{\tau=T_1}=\int_{\underline{\theta}}^{\varphi^{-1}(\underline{x}_{\gamma,\alpha})}\left[e^{-\delta T_1}\varphi(\theta_1)-c\right]f(\theta_2)d\theta_2 \tag{6}$$

$$+\int_{\varphi^{-1}(\underline{x}_{\gamma,\alpha})}^{\varphi^{-1}(\bar{x}_{\gamma,\alpha})}\left[\gamma\left(e^{-\delta T_1}\varphi(\theta_1)+\varphi(\theta_2)\right)-(1+\alpha)c\right]f(\theta_2)d\theta_2$$
$$+\int_{\varphi^{-1}(\bar{x}_{\gamma,\alpha})}^{\bar{\theta}}\left[e^{-\delta T_1}\varphi(\theta_1)+\varphi(\theta_2)-2c\right]f(\theta_2)d\theta_2.$$

Indeed, when $T_1<\overline{T}_{\theta_1}$, we have $e^{-\delta T_1}\varphi(\theta_1)-c>0$ and $e^{-\delta T_1}\varphi(\theta_1)+\varphi(\theta_2)-c>\varphi(\theta_2)-c$. This leads to: (i) an individual service to Agent 1 if $\varphi(\theta_2)<\underline{x}_{\gamma,\alpha}=\frac{(1-\gamma)e^{-\delta T_1}\varphi(\theta_1)}{\gamma}+\frac{\alpha c}{\gamma}$; (ii) a shared service if $\varphi(\theta_2)>\bar{x}_{\gamma,\alpha}=\frac{(1-\alpha)c}{1-\gamma}-e^{-\delta T_1}\varphi(\theta_1)>\underline{x}_{\gamma,\alpha}$; and (iii) separate individual services otherwise.

Then by denoting $\Phi=e^{-\delta T_1}\varphi(\theta_1)$, and by using the fact that $\frac{\partial\Phi}{\partial T_1}=-\delta\Phi$, we get the second derivative of the $\pi(\theta_1)$ with respect to T_1 as:

$$\begin{aligned}\frac{\partial^2\pi(\theta_1)}{\partial(T_1)^2}&=\delta(r+\lambda+\delta)\Phi+\lambda\left[(\Phi-c)\frac{\partial\varphi^{-1}(\underline{x}_{\gamma,\alpha})}{\partial T_1}-\int_{\underline{\theta}}^{\varphi^{-1}(\underline{x}_{\gamma,\alpha})}\delta\Phi f(\theta_2)d\theta_2\right]\\&+\lambda\left[(\gamma(\Phi+\bar{x}_{\gamma,\alpha})-(1+\alpha)c)\frac{\partial\varphi^{-1}(\bar{x}_{\gamma,\alpha})}{\partial T_1}-\left(\gamma(\Phi+\underline{x}_{\gamma,\alpha})-(1+\alpha)c\right)\frac{\partial\varphi^{-1}(\underline{x}_{\gamma,\alpha})}{\partial T_1}-\int_{\varphi^{-1}(\underline{x}_{\gamma,\alpha})}^{\varphi^{-1}(\bar{x}_{\gamma,\alpha})}\gamma\delta\Phi f(\theta_2)d\theta_2\right]\\&+\lambda\left[-(\Phi+\bar{x}_{\gamma,\alpha})-2c)\frac{\partial\varphi^{-1}(\bar{x}_{\gamma,\alpha})}{\partial T_1}-\int_{\varphi^{-1}(\bar{x}_{\gamma,\alpha})}^{\bar{\theta}}\delta\Phi f(\theta_2)d\theta_2\right]\end{aligned}$$

After some algebra, it yields:

$$\frac{\partial^2\pi(\theta_1)}{\partial(T_1)^2}=\delta(r+\lambda+\delta)\Phi-\lambda\int_{\underline{\theta}}^{\bar{\theta}}\delta\Phi f(\theta_2)d\theta_2+\lambda\int_{\varphi^{-1}(\underline{x}_{\gamma,\alpha})}^{\varphi^{-1}(\bar{x}_{\gamma,\alpha})}(1-\gamma)\delta\Phi f(\theta_2)d\theta_2>0.$$

This establishes the convexity of $\pi(\theta_1)$ with respect to T_1 and proves Claim 6. □

Proof of Theorem EC.2. We now know that $T_1(\theta_1)=0$, or $T_1(\theta_1)=\infty$, for each $\theta_1\in[\underline{\theta},\overline{\theta}]$. The next step is to determine for which values of θ_1 it is optimal to set $T_1(\theta)=0$ as opposed to $T_1(\theta)=\infty$. By proceeding as in the proof of Theorem 1, we find that there exists a cutoff $\theta_{\gamma,\alpha}$ such that:

$$T_1(\theta_1)=\begin{cases}0 & \text{if } \theta_1\geq\theta_{\gamma,\alpha},\\ \infty & \text{otherwise.}\end{cases}$$

This completes the proof int he case where $\theta_1>\theta_c$. When $\theta_1\leq\theta_c$, the virtual value of Agent 1 is positive but too small to warrant an individual service. Therefore, Agent 1 is held in queue until Agent 2 arrives, i.e., $T_1(\theta_1)=\infty$. The allocation rule at time τ is identical to the one above. This completes the proof of the theorem. □

3. Details on the benchmarks (Section 4.1)

We provide details on the three posted-prices benchmarks, and compute the six performance metrics considered in the paper. These derivations complement those in EC.1.3 for the optimal mechanism, enabling the performance assessment performed in Section 4.1 of the main paper.

3.1. Benchmark 1: individual services.

The platform provides only individual services at a constant price p. Each agent $i \in \{1,2\}$ purchases the service if and only if $\theta_i \geq p$. The optimal price is $p = \theta_c$. The performance metrics are as follows:

- Probability of service for Agent 1, and for Agent 2: $Pr_1^{\text{Ind}} = Pr_2^{\text{Ind}} = 1 - \theta_c$.
- Expected utility of Agent 1 at time 0 and Agent 2 at time τ:

$$U_1^{\text{Ind}} = U_2^{\text{Ind}} = \int_{\theta_c}^{1} (\theta - \theta_c) d\theta = \frac{(1-\theta_c)^2}{2}$$

- Expected profit of the platform at time 0: $\Pi^{\text{Ind}} = (1-\theta_c)(\theta_c - c)\left(1 + \frac{\lambda}{r+\lambda}\right)$.
- Expected surplus at time 0, including the utility of both agents and the platform's profit:

$$TS^{\text{Ind}} = \left(1 + \frac{\lambda}{r+\lambda}\right) \frac{(1-\theta_c)(1+\theta_c - 2c)}{2}$$

3.2. Benchmark 2: shared services.

The platform charges a fixed price p per shared service to each agent. A service is provided if and only if both agents are available and willing to pay p. We assume that $\lambda > \delta$ to avoid case distinctions, but all arguments can be easily extended otherwise.

For a given price p, Agent 1 is willing to purchase the service at time τ if and only if $e^{-\delta\tau}\theta_1 \geq p$. Under the uniform distribution, this occurs with probability $\left(1 - \frac{p}{e^{-\delta\tau}}\right)$ if $\tau \leq \frac{-\log(p)}{\delta}$, and 0 otherwise. Then, Agent 2 is willing to purchase the service at time τ if and only if $\theta_2 \geq p$, which occurs with probability $(1-p)$. The discounted expected profit at time 0 is given by:

$$\begin{aligned}\pi^S(p) =& (2p-c)\int_0^{\frac{-\log(p)}{\delta}} \lambda e^{-(r+\lambda)\tau}\left(1 - \frac{p}{e^{-\delta\tau}}\right)(1-p)d\tau \\ =& (2p-c)(1-p)\left[\frac{\lambda(1-p^{(r+\lambda)/\delta})}{r+\lambda} - \frac{\lambda p(1-p^{(r+\lambda-\delta)/\delta})}{r+\lambda-\delta}\right]\end{aligned}$$

Let $p_\star = \text{argmin} \pi^S(p)$ be the profit-maximizing price. We compute the performance metrics:

- Probability of service to Agent 1 and 2:

$$Pr_1^{\text{Sh}} = Pr_2^{\text{Sh}} = \int_0^{\frac{-\log(p_\star)}{\delta}} \lambda e^{-\lambda\tau}\left(1 - \frac{p_\star}{e^{-\delta\tau}}\right)(1-p_\star)d\tau = (1-p_\star)\left[\left(1 - p_\star^{\lambda/\delta}\right) - \frac{\lambda p_\star\left(1 - p_\star^{(\lambda-\delta)/\delta}\right)}{\lambda - \delta}\right]$$

- Ex-post expected utility of Agent 1 of θ_1 at time 0, with $\overline{T}_{\theta_1}^S = \max\left\{0, \frac{-\log(\frac{p_\star}{\theta_1})}{\delta}\right\}$ being the latest time when Agent 1 is willing to purchase a service at price $p_\star$. The term $(1-p_\star)$ denotes the probability that Agent 2 is willing to purchase the shared service at time τ; the second term captures Agent 1's expected utility conditional on Agent 2's willingness to purchase.

$$u_1^{\text{Sh}}(\theta_1) = (1-p_\star)\int_0^{\overline{T}_{\theta_1}^S} \lambda e^{-(r+\lambda)\tau}(e^{-\delta\tau}\theta_1 - p_\star)d\tau$$

$$= (1-p_\star)\left[\frac{\lambda(1-e^{-(r+\delta+\lambda)\overline{T}^S_{\theta_1}})\theta_1}{r+\delta+\lambda} - \frac{\lambda(1-e^{-(r+\lambda)\overline{T}^S_{\theta_1}})p_\star}{r+\lambda}\right]$$

We then compute the ex-ante expected utility of Agent 1 as $U_1^{\text{Sh}} = \int_{\underline{\theta}}^{\overline{\theta}} u_1^{\text{Sh}}(\theta_1)d\theta_1$

– Expected utility of Agent 2 at time τ.

$$U_2^{\text{Sh}} = \int_0^{\frac{-\log(p_\star)}{\delta}} \lambda e^{-\lambda\tau}\left(1-\frac{p_\star}{e^{-\delta\tau}}\right)\int_{p_\star}^1(\theta_2-p_\star)d\theta_2 d\tau = \frac{(1-p_\star)^2}{2}\left[\left(1-p_\star^{\lambda/\delta}\right) - \frac{\lambda p_\star\left(1-p_\star^{(\lambda-\delta)/\delta}\right)}{\lambda-\delta}\right]$$

– Expected discounted profit at time 0:

$$\Pi^{\text{Sh}} = (2p_\star - c)(1-p_\star)\left[\frac{\lambda(1-p_\star^{(r+\lambda)/\delta})}{r+\lambda} - \frac{\lambda p_\star(1-p_\star^{(r+\lambda-\delta)/\delta})}{r+\lambda-\delta}\right]$$

– Expected discounted total surplus at time 0:

$$TS^{\text{Sh}} = U_1^{\text{Sh}} + \Pi^{\text{Sh}} + \int_0^{\frac{-\log(p_\star)}{\delta}} \lambda e^{-(r+\lambda)\tau}\left(1-\frac{p_\star}{e^{-\delta\tau}}\right)\int_{p_\star}^1(\theta_2-p_\star)d\theta_2 d\tau$$
$$= U_1^{\text{Sh}} + \Pi^{\text{Sh}} + \frac{(1-p_\star)^2}{2}\left[\frac{\lambda\left(1-p_\star^{(\lambda+r)/\delta}\right)}{\lambda+r} - \frac{\lambda p_\star\left(1-p_\star^{(\lambda+r-\delta)/\delta}\right)}{\lambda+r-\delta}\right]$$

3.3. Benchmark 3: hybrid posted prices for individual and shared services.

The platform charges a price p_I for individual services and a price p_S for shared services. Agents are free to choose their preferred option, but the shared service is only available if both agents are simultaneously available on the platform and both opt for the shared service. Proposition 1 of the main paper identifies a price $\chi(p_I, p_S)$ such that Agent 1 purchases an individual service at $t=0$ if $\theta_1 \geq \chi(p_I, p_S)$. When $\theta_1 < \chi(p_I, p_S)$, Agents 1 and 2 purchase a shared service at time τ if both their willingness to pay at that time exceeds p_S; Agent 1 purchases an individual service if their willingness to pay at that time exceeds p_I and but Agent 2's is less than p_S; and Agent 2 purchases an individual service if their willingness to pay at that time exceeds p_I and but Agent 1's is less than p_S. We prove that proposition below.

Given pair (p_I, p_S), let $\overline{T}^I_{\theta_1}$ (resp., $\overline{T}^S_{\theta_1}$) denote the latest time at which an individual (resp., shared) service yields a positive utility for an agent of type θ_1. Specifically:

$$\overline{T}^I_{\theta_1} = \max\left\{0, \frac{\log\frac{\theta_1}{p_I}}{\delta}\right\}, \quad \overline{T}^S_{\theta_1} = \max\left\{0, \frac{\log\frac{\theta_1}{p_S}}{\delta}\right\}.$$

We can then compute the relevant performance metrics:

– Probability of service for Agent 1, as a function of θ_1. If the agent type exceeds the threshold for individual services, $\chi(p_I, p_S)$, then they are served with probability 1. Otherwise, they are served with probability 1 if Agent 2 arrives before $\overline{T}^I_{\theta_1}$, which occurs with probability

$1 - e^{-\lambda \overline{T}^I_{\theta_1}}$; and they are served with probability $1-p_S$ if Agent 2 arrives between $\overline{T}^I_{\theta_1}$ and $\overline{T}^S_{\theta_1}$, which occurs with probability $e^{-\lambda \overline{T}^I_{\theta_1}} - e^{-\lambda \overline{T}^S_{\theta_1}}$.

$$pr_1^{\text{Hyb}}(\theta_1) = \begin{cases} 1 & \text{if } \theta_1 \geq \chi(p_I,p_S) \\ 1 - e^{-\lambda \overline{T}^I_{\theta_1}} + \left(e^{-\lambda \overline{T}^I_{\theta_1}} - e^{-\lambda \overline{T}^S_{\theta_1}}\right)(1-p_S) & \text{if } \theta_1 < \chi(p_I,p_S) \end{cases}$$

We derive the expected service probability of Agent 1 as $Pr_1^{\text{Hyb}} = \int_{\underline{\theta}}^{\overline{\theta}} pr_1^{\text{Hyb}}(\theta_1) d\theta_1$.

– Expected probability of service for Agent 2, as a function of θ_1. If Agent 1 received an individual service, then Agent 2 is served if and only if their valuation exceeds p_I. Otherwise, if Agent 2 arrives before $\overline{T}^I_{\theta_1}$, they are served if and only if their valuation exceeds p_S; and if Agent 2 arrives after $\overline{T}^S_{\theta_1}$, they are served if and only if their valuation exceeds p_I.

$$pr_2^{\text{Hyb}}(\theta_1) = \begin{cases} 1-p_I & \text{if } \theta_1 \geq \chi(p_I,p_S) \\ \left(1 - e^{-\lambda \overline{T}^S_{\theta_1}}\right)(1-p_S) + e^{-\lambda \overline{T}^S_{\theta_1}}(1-p_I) & \text{if } \theta_1 < \chi(p_I,p_S) \end{cases}$$

We derive the expected service probability of Agent 2 as $Pr_2^{\text{Hyb}} = \int_{\underline{\theta}}^{\overline{\theta}} pr_2^{\text{Hyb}}(\theta_1) d\theta_1$.

– Expected utility of Agent 1 at time 0 as a function of θ_1: an agent of type θ_1 derives a utility of $\theta_1 - p_I$ if Agent 1 is served immediately at price p_I; otherwise, the ex-post utility is equal to:

$$\int_0^{\overline{T}^I_{\theta_1}} \lambda e^{-(r+\lambda)\tau} \left[(1-p_S)(e^{-\delta\tau}\theta_1 - p_S) + p_S(e^{-\delta\tau}\theta_1 - p_I)\right] d\tau + \int_{\overline{T}^I_{\theta_1}}^{\overline{T}^S_{\theta_1}} \lambda e^{-(r+\lambda)\tau}(1-p_S)(e^{-\delta\tau}\theta_1 - p_I) d\tau$$

Therefore, the ex-post utility of Agent 1 of type θ_1 is:

$$u_1^{\text{Hyb}}(\theta_1) = \begin{cases} \theta_1 - p_I & \text{if } \theta_1 \geq \chi(p_I,p_S) \\ \frac{\lambda(1-e^{-(r+\delta+\lambda)\overline{T}^I_{\theta_1}})}{r+\delta+\lambda}\theta_1 - \frac{\lambda(1-e^{-(r+\lambda)\overline{T}^I_{\theta_1}})}{r+\lambda}((1-p_S)p_S + p_s p_I) & \\ +(1-p_S)\left[\frac{\lambda(e^{-(r+\delta+\lambda)\overline{T}^I_{\theta_1}} - e^{-(r+\delta+\lambda)\overline{T}^S_{\theta_1}})}{r+\delta+\lambda}\theta_1 - \frac{\lambda(e^{-(r+\lambda)\overline{T}^I_{\theta_1}} - e^{-(r+\lambda)\overline{T}^S_{\theta_1}})}{r+\lambda}p_S\right] & \text{if } \theta_1 < \chi(p_I,p_S) \end{cases}$$

We then compute the ex-ante expected utility of Agent 1 as $U_1^{\text{Hyb}} = \int_{\underline{\theta}}^{\overline{\theta}} u_1^{\text{Hyb}}(\theta_1) d\theta_1$

– Expected utility for Agent 2 at time τ, as a function of θ_1. These calculations follow a similar logic as those of the probability of service, by replacing the probabilities $1-p$ by the utilities $(1-p)^2/2$ for a price of $p \in \{p_S, p_I\}$.

$$u_2^{\text{Hyb}}(\theta_1) = \begin{cases} \frac{(1-p_I)^2}{2} & \text{if } \theta_1 \geq \chi(p_I,p_S) \\ (1 - e^{-\lambda \overline{T}^S_{\theta_1}})\frac{(1-p_S)^2}{2} + e^{-\lambda \overline{T}^S_{\theta_1}}\frac{(1-p_I)^2}{2} & \text{if } \theta_1 < \chi(p_I,p_S) \end{cases}$$

We then compute the ex-ante expected utility of Agent 2 as $U_2^{\text{Hyb}} = \int_{\underline{\theta}}^{\overline{\theta}} u_2^{\text{Hyb}}(\theta_1) d\theta_1$

– Expected profit, calculated at time 0 as a function of the type of Agent 1: If Agent 1 of type θ_1 receives an immediate service, then the platform derives an immediate profit of $p_I - c$ and a future expected profit of $\frac{\lambda}{r+\lambda}(1-p_I)(p_I-c)$. Otherwise, the platform derives an expected discounted profit of:

$$\int_0^{\overline{T}^I_{\theta_1}} \lambda e^{-(r+\lambda)\tau}\left[(1-p_S)(2p_S-c)+p_S(p_I-c)\right]d\tau + \int_{\overline{T}^I_{\theta_1}}^{\overline{T}^S_{\theta_1}} \lambda e^{-(r+\lambda)\tau}(1-p_S)(2p_S-c)d\tau$$
$$+\int_{\overline{T}^S_{\theta_1}}^{\infty}(1-p_I)(p_I-c)d\tau$$

Therefore, the ex-post profit of the platform is equal to:

$$\pi^{\text{Hyb}}(\theta_1) = \begin{cases} (p_I-c)\left(1+\frac{\lambda}{r+\lambda}(1-p_I)\right) & \text{if } \theta_1 \geq \chi(p_I,p_S) \\ \frac{\lambda(1-e^{-(r+\lambda)\overline{T}^I_{\theta_1}})}{r+\lambda}\left[(1-p_S)(2p_S-c)+p_S(p_I-c)\right] & \\ +\frac{\lambda\left(e^{-(r+\lambda)\overline{T}^I_{\theta_1}}-e^{-(r+\lambda)\overline{T}^S_{\theta_1}}\right)}{r+\lambda}((1-p_S)(2p_S-c)) & \text{if } \theta_1 < \chi(p_I,p_S) \\ +\frac{\lambda(e^{-(r+\lambda)\overline{T}^S_{\theta_1}}}{r+\lambda}((1-p_I)(p_I-c)) & \end{cases}$$

We then compute the ex-ante expected profit as $\Pi^{\text{Hyb}} = \int_{\underline{\theta}}^{\overline{\theta}} \pi^{\text{Hyb}}(\theta_1)d\theta_1$

– Expected discounted total surplus at time 0 as a function of θ_1:

$$ts^{\text{Hyb}}(\theta_1) = \begin{cases} (p_I-c)\left(1+\frac{\lambda}{r+\lambda}(1-p_I)\right)+(\theta_1-p_I)+\frac{\lambda}{r+\lambda}\frac{(1-p_I)^2}{2} & \text{if } \theta_1 \geq \chi(p_I,p_S) \\ u_1^{\text{Hyb}}(\theta_1)+\pi^{\text{Hyb}}(\theta_1)+\frac{\lambda}{r+\lambda}\left((1-e^{(r+\lambda)\overline{T}^S_{\theta_1}})\frac{(1-p_S)^2}{2}+e^{-(r+\lambda)\overline{T}^S_{\theta_1}}\frac{(1-p_I)^2}{2}\right) & \text{if } \theta_1 < \chi(p_I,p_S) \end{cases}$$

We then calculate the expected discounted total surplus at time 0 as $TS^{\text{Hyb}} = \int_{\underline{\theta}}^{\overline{\theta}} ts^{\text{Hyb}}(\theta_1)d\theta_1$.

3.4. Proof of Proposition 1

For any prices p_S and p_I such that $p_I \geq p_S$, we first identify properties of the equilibrium outcome. We then leverage these properties to formulate the platform's profit-maximization problem.

Lemma 2. *If Agent 1 does not purchase an individual service at time $t=0$, then they does not purchase any service until Agent 2 arrives at time $t=\tau$.*

Proof of Lemma 2 Suppose by contradiction that it is optimal for Agent 1 of type θ_1 to purchase an individual service before Agent 2 arrives at some time $t>0$. From their revealed preferences, we know that purchasing an individual service at time t dominates an alternative strategy that consists of waiting an additional infinitesimally small amount of time dt, and, then, purchasing a service at time $t+dt$. This service will be shared (hence, cheaper) in case Agent 2 arrives (which occurs with probability $1-e^{-\lambda dt}$) and $\theta_2 \geq p_S$ (which occurs with probability $1-F(p_S)$), and individual otherwise. This translates into the following inequality:

$$e^{-\delta t}\theta_1 - p_I \geq e^{-rdt}e^{-\lambda dt}(e^{-\delta(t+dt)}\theta_1 - p_I) \\ + e^{-rdt}(1-e^{-\lambda dt})\left[(1-F(p_S))(e^{-\delta(t+dt)}\theta_1 - p_S) + F(p_S)(e^{-\delta(t+dt)}\theta_1 - p_I)\right]$$

We obtain:

$$e^{-\delta t}\theta_1 - p_I \geq e^{-rdt}e^{-\delta(t+dt)}\theta_1 - e^{-rdt}\left(e^{-\lambda dt} + (1-e^{-\lambda dt})(1-F(p_S))\right)p_I - e^{-rdt}(1-e^{-\lambda dt})F(p_S)p_S$$

This yields:

$$e^{-\delta t}\theta_1(1-e^{-(r+\delta)dt}) \geq \left(1-e^{-(r+\lambda)dt} - e^{-(r+\lambda)dt})(1-F(p_S))\right)p_I - e^{-(r+\lambda)dt})F(p_S)p_S$$

But then, by multiplying the left-hand side by $e^{+\delta dt}$ (which is larger than 1), we obtain:

$$e^{-\delta(t-dt)}\theta_1(1-e^{-(r+\delta)dt}) > \left(1-e^{-(r+\lambda)dt} - (1-e^{-(r+\lambda)dt})(1-F(p_S))\right)p_I - (1-e^{-(r+\lambda)dt})F(p_S)p_S.$$

Therefore, purchasing an individual service (before Agent 2 arrives) at $t-dt$ is strictly better than purchasing an individual service (before Agent 2 arrives) at t for Agent 1. This contradicts with the optimality of t, and completes our proof. $\square$

From Lemma 2, we know that Agent 1 either immediately purchases an individual service at $t=0$ or waits until Agent 2 arrives. At time $t=\tau$, they has three options:

- When $e^{-\delta\tau}\theta_1 \geq p_I$, Agent 1 purchases a shared service with Agent 2 if $\theta_2 \geq p_S$. Otherwise, if $\theta_2 < p_S$, they purchases an individual service.
- When $p_S \leq e^{-\delta\tau}\theta_1 < p_I$, Agent 1 receives a shared service if $\theta_2 \geq p_S$. Otherwise, if $\theta_2 < p_S$, they leaves the platform without being served.
- When $e^{-\delta\tau}\theta_1 < p_S$, Agent 1 leaves the platform without being served.

The following lemma shows that Agent 1 purchases an immediate service at $t=0$ if and only if their type θ_1 exceeds a threshold value, denoted by $\chi(p_I, p_S)$.

LEMMA 3. *For any given pair of prices p_S and p_I, there exists a value $\chi(p_I, p_S) > p_I$ such that Agent 1 purchases an immediate service at $t=0$ if and only if $\theta_1 \geq \chi(p_I, p_S)$*

Proof of Lemma 3 If Agent 1 of type θ_1 purchases an immediate service at $t=0$ at price p_I, their utility, denoted by $U_{imm}(\theta_1)$, is given by:

$$U_{imm}(\theta_1) = \theta_1 - p_I$$

It is clear that Agent 1 never purchases an individual service when $\theta_1 < p_I$. We therefore assume that $\theta_1 \geq p_I$. Under this assumption, the times $T^I_{\theta_1}$ and $T^P_{\theta_1}$ given in the proposition are equal to:

$$T^I_{\theta_1} = \frac{1}{\delta}\log\left(\frac{\theta_1}{p_I}\right) \qquad \text{and} \qquad T^P_{\theta_1} = \frac{1}{\delta}\log\left(\frac{\theta_1}{p_S}\right).$$

We now express the expected discounted utility of Agent 1 of type θ_1 resulting from waiting until time $t = \tau$ (when Agent 2 arrives), which we denote by $U_{wait}(\theta_1)$. Recall that:

- If $\tau \leq T^I_{\theta_1}$, then Agent 1 purchases either a shared service or individual service.
- If $T^I_{\theta_1} < \tau \leq T^P_{\theta_1}$, then Agent 1 purchases only a shared service.
- If $\tau > T^P_{\theta_1}$, then Agent 1 does not purchase any service.

Thus, $U_{wait}(\theta_1)$ satisfies:

$$\begin{aligned}
U_{wait}(\theta_1) &= \int_0^{T^I_{\theta_1}} \lambda e^{-(r+\lambda)\tau}\left[(1-F(p_S))(e^{-\delta\tau}\theta_1 - p_S) + F(p_S)(e^{-\delta\tau}\theta_1 - p_I)\right] d\tau \\
&\quad + \int_{T^I_{\theta_1}}^{T^P_{\theta_1}} \lambda e^{-(r+\lambda)\tau}(1-F(p_S))\left(e^{-\delta\tau}\theta_1 - p_S\right) d\tau \\
&= \left[1 - e^{-(r+\delta+\lambda)T^I_{\theta_1}}F(p_S) - e^{-(r+\delta+\lambda)T^P_{\theta_1}}(1-F(p_S))\right]\frac{\lambda}{r+\delta+\lambda}\theta_1 \\
&\quad - \frac{\lambda}{r+\lambda}\left[1 - e^{-(r+\lambda)T^P_{\theta_1}}\right](1-F(p_S))p_S - \frac{\lambda}{r+\lambda}\left[1 - e^{-(r+\lambda)T^I_{\theta_1}}\right]F(p_S)p_I
\end{aligned}$$

Plugging the expressions for $T^I_{\theta_1}$, and $T^P_{\theta_1}$, we obtain:

$$\begin{aligned}
U_{wait}(\theta_1) &= \left(1 - F(p_S)\left(\frac{p_I}{\theta_1}\right)^{\frac{r+\delta+\lambda}{\delta}} - (1-F(p_S))\left(\frac{p_S}{\theta_1}\right)^{\frac{r+\delta+\lambda}{\delta}}\right)\frac{\lambda}{r+\delta+\lambda}\theta_1 \\
&\quad - \frac{\lambda}{r+\lambda}\left(1 - \left(\frac{p_S}{\theta_1}\right)^{\frac{r+\lambda}{\delta}}\right)(1-F(p_S))p_S - \frac{\lambda}{r+\lambda}\left(1 - \left(\frac{p_I}{\theta_1}\right)^{\frac{r+\lambda}{\delta}}\right)F(p_S)p_I
\end{aligned}$$

By defining $\Delta(\theta_1) = U_{imm}(\theta_1) - U_{wait}(\theta_1)$, we have:

$$\begin{aligned}
\Delta(\theta_1) &= \frac{(r+\delta)\theta_1}{r+\delta+\lambda} - p_I + \frac{\lambda}{r+\delta+\lambda}\theta_1^{-\frac{r+\lambda}{\delta}}\left[F(p_S)p_I^{\frac{r+\delta+\lambda}{\delta}} + (1-F(p_S))p_S^{\frac{r+\delta+\lambda}{\delta}}\right] \\
&\quad - \frac{\lambda}{r+\lambda}\left[\theta_1^{-\frac{r+\lambda}{\delta}}\left(F(p_S)p_I^{\frac{r+\delta+\lambda}{\delta}} + (1-F(p_S))p_S^{\frac{r+\delta+\lambda}{\delta}}\right) - (1-F(p_S))p_S - F(p_S)p_I\right] \\
&= \frac{(r+\delta)\theta_1}{r+\delta+\lambda} - \frac{\lambda\delta}{(r+\delta+\lambda)(r+\lambda)}\theta_1^{-\frac{r+\lambda}{\delta}}\left[F(p_S)p_I^{\frac{r+\delta+\lambda}{\delta}} + (1-F(p_S))p_S^{\frac{r+\delta+\lambda}{\delta}}\right] \\
&\quad - p_I + \frac{\lambda}{r+\lambda}\left[(1-F(p_S))p_S + F(p_S)p_I\right],
\end{aligned}$$

which is increasing in θ_1. Thus, for given prices p_I and p_S, there exists a cutoff $\chi(p_I, p_S)$ such that

$$\Delta(\theta_1) = \begin{cases} > 0 & \text{if } \theta_1 > \chi(p_I, p_S) \\ = 0 & \text{if } \theta_1 = \chi(p_I, p_S) \\ < 0 & \text{if } \theta_1 < \chi(p_I, p_S) \end{cases}$$

This completes the proof. $\square$

Proof of Proposition 1. Based on the structure of the equilibrium, we obtain the expected profit of the platform for given values of p_I and p_S:

$$\begin{aligned}
\Pi(p_I, p_S) = & [1 - F(\chi(p_I, p_S))] \left[(p_I - c) + \frac{\lambda}{r+\lambda}(1 - F(p_I))(p_I - c) \right] \\
& + \int_{\underline{\theta}}^{\chi(p_I, p_S)} \int_0^{T_{\theta_1}^I} \lambda e^{-(r+\lambda)\tau} \left[(1 - F(p_S))(2p_S - c) + F(p_S)(p_I - c) \right] d\tau f(\theta_1) d\theta_1 \\
& + \int_{\underline{\theta}}^{\chi(p_I, p_S)} \int_{T_{\theta_1}^I}^{T_{\theta_1}^P} \lambda e^{-(r+\lambda)\tau} \left[(1 - F(p_S))(2p_S - c) \right] d\tau f(\theta_1) d\theta_1 \\
& + \int_{\underline{\theta}}^{\chi(p_I, p_S)} \int_{T_{\theta_1}^P}^{\infty} \lambda e^{-(r+\lambda)\tau} \left[(1 - F(p_I))(p_I - c) \right] d\tau f(\theta_1) d\theta_1.
\end{aligned}$$

Then the platform's problem is defined as:

$$\max_{p_I, p_S} \quad \Pi(p_I, p_S)$$

This completes the proof of Proposition 1. $\square$